HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12

MATHEMATICS

Half Yearly Examination Term 1 2010
HSC COURSE

ASSESSMENT TASK 2

EXAMINERS ~ D. CRANCHER, S. HACKETT, P. BICZO, S. FAULDS, J. DILLON

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – 2 hours.
- Attempt all questions.
- All questions are of equal value and are not necessarily arranged in order of difficulty.
- All necessary working should be shown in every question.
- This paper contains ten (10) questions.
- Total Marks – 80 marks
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators and mathematical templates may be used.
- Each question is to be started in a new Answer Booklet.
- This examination paper must **NOT** be removed from the examination centre.

STUDENT NAME AND NUMBER: ______________________________

TEACHER: ______________________________

Question 1

(a) Evaluate **(2 marks)**

$$\sqrt{\frac{275\cdot 4}{5\cdot 2\times 3\cdot 9}}$$

correct to two significant figures.

(b) Express **(2 marks)**

$$\frac{(2x-3)}{2}-\frac{(x-1)}{5}$$

as a single fraction in its simplest form.

(c) Solve

$$3-2x\geq 7$$ **(2 marks)**

(d) Find the integers a and b such that:

$$\left(5-\sqrt{2}\right)^2=a+b\sqrt{2}$$ **(2 marks)**

Question 2

(a) Differentiate $(4x+3)(2x^3-5)$ with respect to x **(2 marks)**

(b) Differentiate the following functions:

(i) **(2 marks)**

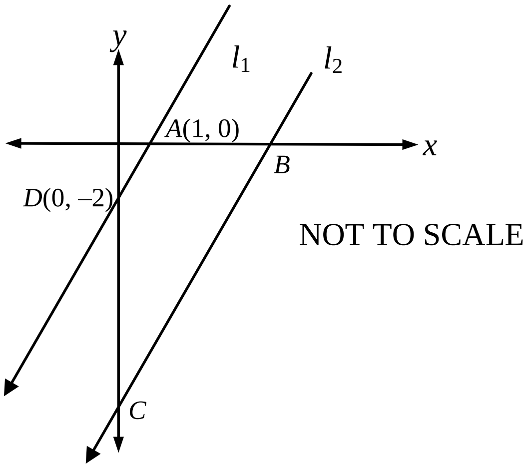
$$y=\frac{2x}{x^2+1}$$

(ii) **(2 marks)**

$$f(x)=\left(3x^2+4\right)^5$$

(c) Find $f'(2)$ if $f(x)=x^4+5x^{-1}$ **(2 marks)**

Question 3



In the diagram, the line l_1 passes through the points $A(1, 0)$ and $D(0, -2)$.
The line l_2 is parallel to l_1 and passes through the point $(5, -2)$.

(a) Write down the equation of the line l_1 in the form $y = mx + b$. **(1 mark)**

(b) Show that the equation of the line l_2 is: **(2 marks)**

$$2x - y - 12 = 0$$

(c) Calculate, in **exact** form, the perpendicular distance between the point $A(1, 0)$ and the line l_2. **(2 marks)**

(d) Find the length of AD. **(1 mark)**

(e) Given $BC = 5\sqrt{5}$ units, calculate the area of the trapezium $ABCD$. **(2 marks)**

Question 4

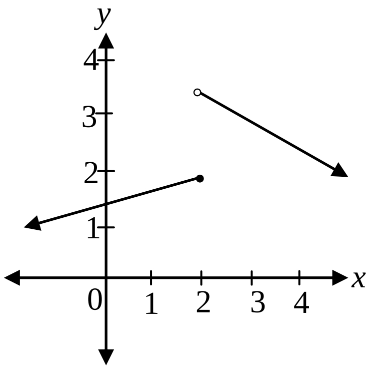
(a) Evaluate

(i) $\lim_{x \to 3} \dfrac{x^2 - 2x - 3}{x - 3}$ **(2 marks)**

(ii) $\lim_{x \to \infty} \dfrac{3x^2 + 1}{x^2 - 5x}$ **(1 mark)**

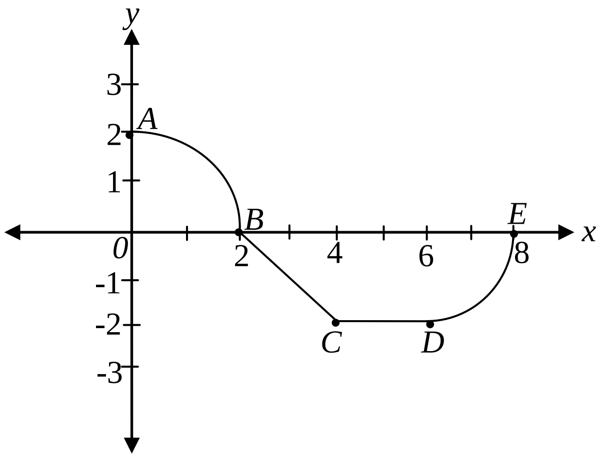
(b) Find the co-ordinates of the point on $f(x) = x^2 + 4x - 9$ at which the tangent is parallel to the x axis. **(2 marks)**

(c)



Is this function continuous? Give a reason. **(1 mark)**

(d)



For what values of x satisfying $0 < x < 8$ is the function f NOT differentiable ? **(2 marks)**

Question 5

(a) Write the equation of the parabola with vertex at the origin, axis of symmetry the *y* axis and passing through the point $(-4,8)$ **(2 marks)**

(b) A parabola has equation $x^2 = 8y$. The tangent at the point $A(4,2)$ meets the directrix at Q.

(i) Draw a diagram showing this information **(1 mark)**

(ii) Find the co-ordinates of Q. **(2 marks)**

(c) For the parabola $x^2 - 6x + 41 = 8y$, find:

(i) the focal length **(2 mark)**

(ii) the coordinates of the vertex **(1 mark)**

Question 6

(a) Solve the quadratic equation **(2 marks)**

$$3x^2 = 5x - 2$$

(b) Solve the inequality: **(2 marks)**

$$x^2 - 4x > 0$$

(c) Show the quadratic equation **(2 marks)**

$$3x^2 - 23x + 1 = 0$$

has two unequal real and irrational roots.

(d) Find the value (s) of *m* for which the equation **(2 marks)**

$$4x^2 - mx + 9 = 0$$

has exactly one real root.

Question 7

a) The first term of an arithmetic series is 9 and the fourth term is 27. Find

(i) the common difference **(2 marks)**

(ii) the sum of the first 20 terms. **(2 marks)**

b) An infinite geometric series has a limiting sum of 24. **(2 marks)**
If the first term is 15, find the common ratio.

c) There are 15 apples in a row, 2 metres apart. The first apple is 2 metres from a basket. How far does a boy run who starts at the basket and returns the apples to the basket one by one? **(2 marks)**

Question 8

(a) Draw a neat sketch of the locus of a point $P(x, y)$ which moves on the Cartesian Plane so that $y > x^2$. **(2 marks)**

(b) $M(-4, -1)$ and $N(2, 7)$ are two fixed points on the Cartesian Plane. $P(x, y)$ is a point that moves so that $PM \perp PN$.

(i) Write down the condition for a pair of lines or intervals to be perpendicular. **(1 mark)**

(ii) Use your answer from part (i) and the co-ordinates of M, N and P to show that the equation of the locus of P is : **(3 marks)**

$$(x+1)^2 + (y-3)^2 = 25$$

(iii) The locus of P is a circle. State the centre and radius of the circle. **(2 marks)**

Question 9

(a) Let α and β be the roots of the equation $x^2 - 5x + 2 = 0$.

Find the values of

(i) $\alpha + \beta$ **(1 mark)**

(ii) $\alpha\beta$ **(1 mark)**

(iii) $\alpha^2 + \beta^2$ **(2 marks)**

(b) For what values of p is the expression **(2 marks)**

$$x^2 - 3x + 2p - 1$$

positive for all real values of x?

(c) Solve the equation **(2 marks)**

$$3^{2x} + 2 \times 3^x - 15 = 0$$

Question 10

An investor wants to borrow $1 000 000 to purchase a block of units at Penrith from *Bank X* which offers an interest rate of 6% p.a. monthly reducible.
The investor is to repay the loan in equal monthly instalments *M*, over 10 years.

(a) If A_n is the amount owing after n instalments, develop expressions for A_1, A_2, A_3 and show that: **(3 marks)**

$$A_n = 1000000(1.005)^n - M(1.005^{n-1} + \ldots + 1.005^2 + 1.005 + 1)$$

(b) Hence show that the monthly instalment, M is given by: **(2 marks)**

$$M = \frac{5000(1.005)^{120}}{1.005^{120} - 1}$$

(c) Calculate the value of the monthly instalment, M, to the nearest cent. **(1 mark)**

(d) Determine the amount still owing to *Bank X* after 5 years, to the nearest cent. **(2 marks)**

[End of exam]

Year 12	Mathematics	Half Yearly Examination 2010
Question No. 1	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

Outcome	Solutions	Marking Guidelines
P3	1. (a) $\sqrt{\dfrac{275\cdot 4}{5\cdot 2\times 3\cdot 9}}$ $= 3\cdot 685089098...$ $= 3\cdot 7$(to 3 significant figures)	2 marks awarded for complete correct solution 1 mark awarded for correct calculation but incorrectly rounded to 2 significant figures.
P3, P4	(b) $\dfrac{(2x-3)}{2}-\dfrac{(x-1)}{5}$ $=\dfrac{5(2x-3)-2(x-1)}{10}$ $=\dfrac{10x-15-2x+2}{10}$ $=\dfrac{8x-13}{10}$	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution
P4	(c) $3-2x\geq 7$ $-2x\geq 4$ $x\leq -2$	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution
P3	(d) $\left(5-\sqrt{2}\right)^2 = a+b\sqrt{2}$ $25-10\sqrt{2}+2 = a+b\sqrt{2}$ $27-10\sqrt{2} = a+b\sqrt{2}$ $\therefore a=27$ and $b=-10$	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution

Year 12	Mathematics	Half Yearly Examination 2010
Question No. 2	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

P7 determines the derivative of a function through routine application of the rules of differentiation
P8 understands and uses the language and notation of calculus

Outcome	**Solutions**	**Marking Guidelines**
P7	2. (a) $(4x+3)(6x^2)+(2x^3-5)(4)$ or equivalent answer.	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution
P7	(b) (i) $\dfrac{dy}{dx}=\dfrac{(x^2+1)(2)-(2x)(2x)}{(x^2+1)^2}$ or equivalent answer.	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution
P7	(ii) $f'(x)=5(3x^2+4)^4(6x)$	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution
P7, P8	(c) $f'(x)=4x^3-5x^{-2}$ $f'(2)=\dfrac{123}{4}$	2 marks awarded for complete correct solution 1 mark awarded for partial correct solution

Outcomes Addressed in this Question

H5 applies appropriate techniques from the study of calculus, **geometry**, probability, trigonometry and series to solve problems

Outcome	Solutions	Marking Guidelines
H5	**3.(a)** l_1: $y = 2x - 2$	**1 mark** Correct answer
H5	**(b)** l_2: $m = 2$ (since $l_1 \parallel l_2$) passes through $(5, -2)$ $y - y_1 = m(x - x_1)$ $y + 2 = 2(x - 5)$ $y + 2 = 2x - 10$ $2x - y + 12 = 0$ as required	**2 marks** Correct solution **1 mark** Correctly states gradient of required line and point-gradient form of equation of straight line.
H5	**(c)** $d = \frac{\lvert Ax_1 + By_1 + C\rvert}{\sqrt{A^2 + B^2}}$ $= \frac{\lvert 2 \times 1 - 1 \times 0 - 12\rvert}{\sqrt{2^2 + 1^2}}$ $= \frac{\lvert -10\rvert}{\sqrt{5}}$ $= \frac{10}{\sqrt{5}}$ $= 2\sqrt{5}$ units	**2 marks** Correct solution (not necessary to rationalise denominator) **1 mark** Correctly states perpendicular distance formula and makes substantial progress towards a correct solution.
H5	**(d)** Using Pythagoras' Theorem in ΔADO $AD^2 = 2^2 + 1^2$ $\therefore AD = \sqrt{5}$	**1 mark** Correct answer
H5	**(e)** Area of Trapezium ABCD $= \frac{h}{2}(a + b)$ $= \frac{2\sqrt{5}}{2}\left(\sqrt{5} + 5\sqrt{5}\right)$ $= \sqrt{5} \times 6\sqrt{5}$ $= 30$ units^2 Note: Typographical error on examination paper. Distance BC = $6\sqrt{5}$ units, not $5\sqrt{5}$ as stated. This gives an area of 35 units^2. Both answers accepted as correct.	**2 marks** Correct solution **1 mark** Substantial progress towards correct solution including area formula for trapezium.

Year 12	Mathematics Task 2	2010
Question No. 4	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

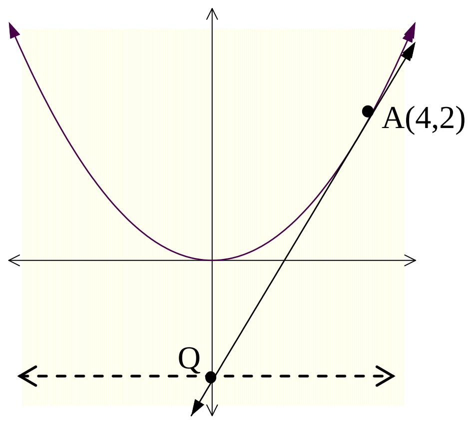
P4 Chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques
P5 Understands the concept of a function and the relationship between a function and its graph
P6 Relates the derivative of a function to the slope of its graph
P7 Determines the derivative of a function through routine application of the rules of differentiation

Outcome	Solutions	Marking Guidelines
P4	4. a) (i) $\lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \lim_{x\to 3} \frac{(x+1)(x-3)}{x-3}$ $= \lim_{x\to 3} (x+1)$ $= 3+1 = 4$	2 marks: correct solution 1 mark: partially correct solution
P4	(ii) $\lim_{x\to\infty} \frac{3x^2+1}{x^2-5x} = \lim_{x\to\infty} \frac{3+\frac{1}{x^2}}{1-\frac{5}{x}}$ $= \frac{3+0}{1-0} = 3$	1 mark: correct solution
P6, P7 P4	b) $f(x) = x^2+4x-9$ $f'(x) = 2x+4$ Tangent parallel to x axis when gradient = 0. $\therefore 2x+4=0$ $\therefore 2x=-4 \quad \therefore x=-2$ When $\therefore x=-2$, $y=(-2)^2+4\times -2-9=-13$ Co-ordinates are $(-2,-13)$	2 marks: correct solution 1 mark: partially correct solution
P5	c) Not continuous at $x=2$ as there is a gap in the graph.	1 mark: correct answer and explanation
P6	d) Not differentiable if there is a sharp corner. At B and C this occurs. $\therefore$ at $x=2$ and $x=4$ not differentiable.	2 marks: correct solution 1 mark: partially correct solution

Year 12	Mathematics Task 2	2010
Question No. 5	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

P4 Chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric Techniques
P5 Understands the concept of a function and the relationship between a function and its graph
P6 Relates the derivative of a function to the slope of its graph

Outcome	Solutions	Marking Guidelines
P4	5. a) Equation in form $x^2 = 4ay$ as must be concave up. $(-4,8)$ lies on it. $\therefore 16 = 4.a.8$ $\therefore a = \frac{1}{2}$ $\therefore x^2 = 2y$ is the equation	2 marks: correct answer 1 mark: correctly finds *a* or equivalent
P5	b) (i) 	1 mark: correctly marks given information
P4, P6	(ii) Since $x^2 = 8y$, $y = \frac{x^2}{8}$ and $4a = 8$, $a = 2$ $\therefore \frac{dy}{dx} = \frac{2x}{8}$. At $x = 4$, $\frac{dy}{dx} = \frac{2\times 4}{8} = 1$ Equation of tangent is $y - 2 = 1(x-4)$ using $y - y_1 = m(x - x_1)$ $\therefore$ tangent is $y = x - 2$ Tangent meets directrix $y = -2$ at Q. $\therefore -2 = x - 2$. $x = 0$. $\therefore Q(0,-2)$	2 marks: correct answer 1 mark: partially correct solution
P4	c) (i) $x^2 - 6x + 41 = 8y$ $x^2 - 6x = 8y - 41$ $x^2 - 6x + 9 = 8y - 32$ $(x-3)^2 = 8(y-4)$ which is in the form $(x-h)^2 = 4a(y-k)$ with $4a = 8$ $\therefore$ focal length is $a = 2$ units	2 marks: correct answer from correct method
P4	(ii) vertex $(3,4)$	1 mark: correct answer or equivalent

Year 12 Question No. 6	Mathematics Solutions and Marking Guidelines	Half Yearly Examination 2010

Outcomes Addressed in this Question

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

H2 constructs arguments to prove and justify results

Outcome	Solutions	Marking Guidelines
P4	6. (a) $3x^2 = 5x - 2$ $3x^2 - 5x + 2 = 0$ $(3x-2)(x-1) = 0$ $\therefore x = \frac{2}{3}$ or 1	**Award 2** ~ correct answers **Award 1** ~ attempts to use appropriate method to solve the equation
P4	(b) $x^2 - 4x > 0$ $x(x-4) > 0$ $\therefore\ x < 0$ or $x > 4$	**Award 2** ~ correct answers **Award 1** ~ correct factorisation **or** attempts to solve inequality by an appropriate method.
P4, H2	(c) $\Delta = b^2 - 4ac$ $= (-23)^2 - 4.3.1$ $= 517$ Since $\Delta > 0 \Rightarrow$ two unequal roots. And since Δ is not a perfect square $\Rightarrow$ the roots are irrational.	**Award 2** ~ correct solution **Award 1** ~ insufficient justification provided.
P4, H2	(d) $\Delta = b^2 - 4ac$ $= (-m)^2 - 4.4.9$ $= m^2 - 144$ One real root $\Rightarrow\ \Delta = 0$ $\therefore\ m^2 - 144 = 0$ $\therefore\ m = \pm 12$	**Award 2** ~ correct solution **Award 1** ~ substantial progress towards solution.

Year 12 Question No. 7	Mathematics Solutions and Marking Guidelines	Half Yearly Examination 2010

Outcomes Addressed in this Question

H5 – applies appropriate techniques from the study of series to solve problems

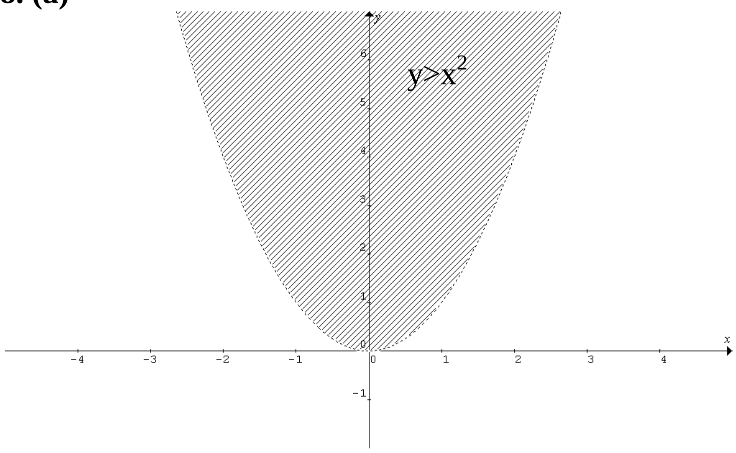
Outcome	Solutions	Marking Guidelines
H5	7. a)i) $a = 9, T_4 = 27$ $a + 3d = 27$ $9 + 3d = 27$ $3d = 18$ $d = 6$	**2 marks** – correct solution **1 mark** – substantially correct solution
H5	a)ii) $S_{20} = \frac{20}{2}(2(9) + 19(6))$ $= 1320$	**2 marks** – correct solution **1 mark** – substantially correct solution
H5	b) $S_\infty = 24, a = 15$ $S_\infty = \frac{a}{1-r}$ $24 = \frac{15}{1-r}$ $24 - 24r = 15$ $9 = 24r$ $r = \frac{9}{24}$ $r = \frac{3}{8}$	**2 marks** – correct solution **1 mark** – substantially correct solution
H5	c) $Dist = 4 + 8 + 12 + ... + 60$ Arithmetic series, $a = 4, d = 4, n = 15$ $Dist = \frac{15}{2}(2(4) + 14(4))$ $= 480 metres$	**2 marks** – correct solution **1 mark** – substantially correct solution

Year 12 Mathematics Half Yearly Examination 2010

Question No. 8 Solutions and Marking Guidelines

Outcomes Addressed in this Question

H5 applies appropriate techniques from the study of calculus, **geometry**, probability, trigonometry and series to solve problems

Outcome	Solutions	Marking Guidelines
H5	**8. (a)** 	**2 marks** Graph correctly drawn. **1 mark** Graph substantially correct with a single error in shading or boundary
H5	**(b) (i)** A pair of lines will be perpendicular if the product of their gradients is –1. ie. $m_1 \times m_2 = -1$	**1 mark** Correct answer
H5	**(ii)** $m_{PM} = \frac{y+1}{x+4}$ $m_{PN} = \frac{y-7}{x-2}$ Equation of locus: $m_{PM} \times m_{PN} = -1$ $\frac{y+1}{x+4} \times \frac{y-7}{x-2} = -1$ $(y+1)(y-7) = -(x+4)(x-2)$ $y^2 - 6y - 7 = -x^2 - 2x + 8$ $x^2 + 2x + y^2 - 6y = 15$ $x^2 + 2x + 1 + y^2 - 6y + 9 = 15 + 9 + 1$ $(x+1)^2 + (y-3)^2 = 25$	**3 marks** Correct solution using result from (i) **2 marks** Substantial progress towards correct solution using result from (i) **1 mark** Determines expressions for gradients from given information and connects these using result in (i)
H5	**(iii)** The locus of point P is a circle. Centre(–1, 3) Radius = 5	**2 marks** Correct answer for both centre and radius **1 mark** One of centre or radius stated correctly

Year 12 Question No. 9	Mathematics Solutions and Marking Guidelines	Half Yearly Examination 2010

Outcomes Addressed in this Question

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

H2 constructs arguments to prove and justify results

Outcome	Solutions	Marking Guidelines
	9.	
(a) P4	(i) $\alpha+\beta=-\frac{-5}{1}=5$	**Award 1** ~ correct answer
	(ii) $\alpha\beta=\frac{2}{1}=2$	**Award 1** ~ correct answer
	(iii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=5^2-2.2=21$	**Award 2** ~ correct answers **Award 1** ~ attempts to use an appropriate method.
(b) P4, H2	Since $a=1$, we want $\Delta<0$ $\Delta=b^2-4ac=(-3)^2-4.1.(2p-1)$ $=13-8p$ $\therefore\ 13-8p<0$ $\therefore\ p>\frac{13}{8}$	**Award 2** ~ correct solution **Award 1** ~ substantial progress towards solution.
(c) P4	$3^{2x}+2\times3^x-15=0$ $(3^x)^2+2\times3^x-15=0$ Let $u=3^x$, $u^2+2u-15=0$ $(u+5)(u-3)=0$ $\therefore\ u=-5$ or 3 $\therefore\ 3^x=-5$ or 3 $\therefore\ x=1$ (only valid solution)	**Award 2** ~ correct solution **Award 1** ~ substantial progress towards solution.

Year 12	Mathematics	Half Yearly Examination 2010
Question No.10	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

H5 – applies appropriate techniques from the study of series to solve problems

Outcome	Solutions	Marking Guidelines
H5	10. a) $6\%\ p.a. = 0.005\ per\ month$ Let A_n = the amount owing after n instalments $A_1 = 1000000(1.005) - M$ $A_2 = A_1(1.005) - M$ $= (1000000(1.005) - M)(1.005) - M$ $= 1000000(1.005)^2 - M(1.005 + 1)$ $A_3 = A_2(1.005) - M$ $= (1000000(1.005)^2 - M(1.005 + 1))(1.005) - M$ $= 1000000(1.005)^3 - M(1.005^2 + 1.005 + 1)$ $A_n = 1000000(1.005)^n - M(1.005^{n-1} + ... + 1.005^2 + 1.005 + 1$	**<u>3 marks</u>** – correct solution for all parts **<u>2 marks</u>** – substantially correct solution **<u>1 mark</u>** – some progress towards correct solution
H5	b) $After\ n\ instalments, A_n = 0$ $After\ 10\ years\ n = 10 \times 12 = 120.$ $\therefore 0 = 1000000(1.005)^{120} - M(1.005^{119} + ... + 1.005^2 + 1.005 + 1)$ $\therefore M = \dfrac{1000000(1.005)^{120}}{1.005^{119} + ... + 1.005^2 + 1.005 + 1}$ $1.005^{119} + ... + 1.005^2 + 1.005 + 1$ is a geometric series, $a = 1, r = 1.$ $\therefore M = \dfrac{1000000(1.005)^{120}}{1(\frac{1.005^{120} - 1}{0.005})}$ $\therefore M = \dfrac{5000(1.005)^{120}}{1.005^{120} - 1}$	**<u>2 marks</u>** – correct solution **<u>1 mark</u>** – substantially correct solution
H5	c) $M = \$11102.05$(to the nearest cent)	**<u>1 mark</u>** – correct solution
H5	d) After 5 years, $n = 5 \times 12 = 60$ $A_{60} \approx 1000000(1.005)^{60} - 11102.05(1.005^{59} + ... + 1.005^2 + 1.0$ $= 1000000(1.005)^{60} - 11102.05(\frac{1.005^{60} - 1}{0.005})$ $= \$574\ 259.79$ (to nearest cent)	**<u>2 marks</u>** – correct solution **<u>1 mark</u>** – substantially correct solution